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Generalized Carleson Operator and Convergence of Walsh Type Wavelet Packet Expansions

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Abstract
In this paper, two new theorems on generalized Carleson Operator for a Walsh type wavelet packet system and for periodic Walsh type wavelet packet expansion of a function $f \epsilon L^p[0,1)$, $1 < p < \infty$, have been established.
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I. Introduction

A new class of orthogonal expansion in $L^2(\mathbb{R})$ with good time frequency and regularity approximation properties are obtained in wavelet analysis.

These expansions are useful in Signal analysis, Quantum mechanics, Numerical analysis, Engineering and Technology. Wavelet analysis generalizes an orthogonal wavelet expansion with suitable time frequency properties. In transient as well as stationary phenomena, this approach have applications over wavelet and short-time Fourier analysis. The properties of orthonormal bases have been studied in $L^2(R)$ for wavelet expansion. The basic wavelet packet expansions of L^p-functions, $1 < p < \infty$, defined on the real line and the unit interval have significant importance in wavelet analysis. Walsh system is an example of a system of basic stationary wavelet packets (Billard [1] and Sjolin [9]). Paley [7], Billard [1] Sjolin [9] and Nielsen [6] have investigated point wise convergent properties of Walsh expansion of given $L^p[0,1)$ functions. At first, in 1966, Lennart Carleson [3] introduced an operator which is presently known as Carleson operator. In this paper, generalized Carleson operators for Walsh type wavelet packet expansion and for periodic Walsh type wavelet packet expansion for any $f \in L^p[0,1)$ are introduced.

Two new theorems have been established

(1) The generalized Carleson operator for any Walsh-type wavelet packet system is of strong type (p,p), $1 < p < \infty$.
(2) The generalized Carleson operator for periodic Walsh-type wavelet packet expansion for a function $f \in L^p(\mathbb{R})$, $1 < p < \infty$, converges a.e..

II. Definitions and Preliminaries

The structure in which non-stationary wavelet packets live is that of a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ .To every multiresolution analysis we have an associated scaling function ϕ and a wavelet ψ with the properties that

$$V_j = \overline{span}\left\{2^{\frac{j}{2}}\phi(2^j.-k); k\epsilon\mathbb{Z}\right\} \text{ and } \left\{\psi_{j,k} \equiv 2^{\frac{j}{2}}\psi(2^j.-k); j,k\epsilon\mathbb{Z}\right\}$$

are an orthonormal basis for $L^2(\mathbb{R})$. Write $W_j = \overline{span}\left\{2^{\frac{j}{2}}\psi(2^j.-k); k\epsilon\mathbb{Z}\right\}$.

Let $\mathbb{N}$ be the set of natural number and $\left(F_0^{(p)}, F_1^{(p)}\right)$, $p\,\epsilon\mathbb{N}$, be a family of bounded operators on $l^2(\mathbb{Z})$ of the form

$$\left(F_\epsilon^{(p)}a\right)_k = \sum_{n\in\mathbb{Z}} a_n \; h_\epsilon^p(n-2k), \qquad \epsilon = 0,1$$

with $h_1^{(p)}(n) = (-1)^n\, h_0^{(p)}(1-n)$ a sequence in $l^1(\mathbb{Z})$ such that

$$F_0^{(p)*}F_0^{(p)} + F_1^{(p)*}F_1^{(p)} = 1 \qquad\qquad F_0^{(p)}F_1^{(p)*} = 0.$$

We define the family of functions $\{w_n\}_0^\infty$ recursively by letting $w_0 = \phi$, $w_1 = \psi$ and then for $n\epsilon\mathbb{N}$

$$w_{2n}(x) = 2\sum_{q\in\mathbb{Z}} h_0^{(p)}(q) w_n(2x-q) \tag{2.1}$$

$$w_{2n+1}(x) = 2\sum_{q\in\mathbb{Z}} h_1^{(p)}(q) w_n(2x-q) \tag{2.2}$$

where $2^p \leq n < 2^{p+1}$. The family $\{w_n\}_0^\infty$ is our basic non –stationary wavelet packet.
It is known that

$$\{w_n(.-k); n \geq 0, k \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$.Moreover,$\{w_n(.-k); 2^j \leq n < 2^{j+1}, k \in \mathbb{Z}\}$ is an orthonormal basis for $W_j = \overline{span}\left\{2^{\frac{j}{2}} w_1(2^j.-k); k \in \mathbb{Z}\right\}$.

Each pair $\left(F_0^{(p)}, F_1^{(p)}\right)$can be chosen as a pair of quardrature mirror filters associated with a multiresolution analysis, but this is not necessary.
The trigonometric polynomial given by

$$m_0^{(p)}(\xi) = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} h_0^{(p)}(k)e^{-ik\xi} \text{ and } m_1^{(p)}(\xi) = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} h_1^{(p)}(k)e^{-ik\xi}$$

are called the symbols of filters.

The Haar low- pass quadrature mirror filter$\{h_0(k)\}_k$ is given by $h_0(0) = h_0(1) = \frac{1}{\sqrt{2}}, h_0(k) = 0$ otherwise, and the associate high-pass filter $\{h_1(k)\}_k$ is given by $h_1(k) = (-1)^k\, h_0(1-k)$.

2.1 Walsh function and their properties

The Walsh system $\{W_n\}_{n=0}^\infty$ is defined recursively on $[0,1)$ on considering

$$W_0(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & otherwise \end{cases}$$

and

$$W_{2n}(x) = W_n(2x) + W_n(2x-1),$$
$$W_{2n+1}(x) = W_n(2x) - W_n(2x-1).$$

Observe that the Walsh system is the family of wavelet packets obtained by considering $\varphi = W_0$

$$\psi(x) = \begin{cases} 1, & 0 \leq x < \frac{1}{2}; \\ -1, & \frac{1}{2} \leq x < 1; \\ 0, & otherwise. \end{cases}$$

and using the Haar filters in the definition of the non- stationary wavelet packets.
The Walsh system is closed under point wise multiplication. Define the binary operator $\oplus: \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ by m$\oplus\, n = \sum_{i=0}^\infty |m_i - n_i| 2^i$, where $m = \sum_{i=0}^\infty m_i 2^i, n = \sum_{i=0}^\infty n_i 2^i$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then

$$W_m(x)W_n(x) = W_{m\oplus n}(x), \qquad (2.3)$$

(Schipp et al.[8]).

We can carry over the operator $\oplus$ to the interval [0,1] by identifying those x$\in$ [0,1] with a unique expansion x=$\sum_{j=0}^\infty x_j\, 2^{-j-1}$ (almost all x$\in$ [0,1] has such a unique expansion) by there associated binary sequence $\{x_i\}$. For two such point $x, y \in [0,1]$, define

$$x\oplus\, y = \sum_{j=0}^\infty |x_i - y_i| 2^{-j-1}.$$

The operation $\oplus$ is defined for almost all $x, y \in [0,1]$.Using this definition we have

$$W_n(x \oplus y) = W_n(x)W_n(y) \qquad (2.4)$$

for every pair x,y for which $x \oplus y$ is defined (Golubov et al.[4]).

2.2 Walsh type wavelet packets

Let $\{w_n\}_{n\geq 0, k\in\mathbb{Z}}$ be a family of non-stationary wavelet packets constructed by using of family $\left\{h_0^{(p)}(n)\right\}_{p=0}^\infty$ of finite filters for which there is a constant$K \in \mathbb{N}$ such that $h_0^{(p)}(n)$ is the Haar filter for every $p \geq K$.
If $w_1 \in C^1(\mathbb{R})$ and it has compact support then we call $\{w_n\}_{n>0}$ a family of Walsh type wavelet packets.

2.3 Periodic Walsh type wavelet packet

As Meyer[6-a], an orthonormal basis for $L^2[0,1)$ is obtained periodizing any orthonormal wavelet basis associated with a multiresolution analysis. The periodization works equally well with non stationary wavelet packets.

Let$\{w_n\}_{n=0}^\infty$ be a family of non-stationary basic wavelet packets satisfying $|w_n(x)| \leq C_n(1+|x|)^{-1-\epsilon_n}$ for some $\epsilon_n > 0$, n$\in \mathbb{N}_0$. For n$\in \mathbb{N}_0$ we define the corresponding periodic wavelet packets $\widetilde{w}_n$ by

$$\widetilde{w}_n(x) = \sum_{k\in\mathbb{Z}} w_n(x-k).$$

It is important to note that the hypothesis about the point- wise decay of the wavelet packets w_n ensures that the periodic wavelet packets are well defined in $L^p[0,1)$ for $1 \leq p \leq \infty$. Wickerhauser and Hess[11] have proved that, the family $\{\widetilde{w}_n\}_{n=0}^\infty$ is an orthonormal basis for $L^p[0,1)$.

2.4 (C,1) and (C,1,1) method

The series

$$\sum_{n=0}^{\infty} a_n = a_0+a_1+a_2+\cdots,$$

is said to be convergent to the sum S. If the partial sum

$$S_n = a_0+a_1+a_2+\cdots+a_n$$

tends to finite limit S when $n\to\infty$; and a series which is not convergent is said to be divergent.

If

$$S_n = a_0+a_1+a_2+\cdots+a_n,$$

and

$$\lim_{n\to\infty}\frac{S_0+S_1+S_2+\cdots+S_n}{n+1} = S,$$

Then we call S the (C,1) sum of $\sum a_n$ and the (C,1) limits of $S_{n.}$ (Hardy[5],p.7)

The series1+0-1+1+0-1+1+0...,is not convergent but it is summable (C,1) to the sum$\frac{2}{3}$, (Titchmarsh [10], p.4111).

$$\text{Let } \sum_{m=0}^{\infty}\sum_{n=o}^{\infty} a_{m,n} = \begin{matrix} a_{0,0}+a_{0,1}+a_{0,2}+\cdots \\ +a_{1,0}+a_{1,1}+a_{1,2}+\cdots \\ +a_{2,0}+a_{2,1}+a_{2,2}+\cdots \end{matrix}$$

be a double infinite series (Bromwich[2],p.92). The partial sum of double infinite series denoted by$S_{m,n}$, and defined by

$$S_{m,n} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j}\,.$$

Write

$$\sigma_{m,n} = \frac{1}{(m+1)(n+1)}\sum_{i=1}^{m}\sum_{j=1}^{n} S_{i,j}$$

$$=\sum_{i=1}^{m}\sum_{j=1}^{n}\left(1-\frac{i}{m+1}\right)\left(1-\frac{j}{n+1}\right)a_{i,j}\,. \quad (2.5)$$

If $\sigma_{m,n}\to S$ as $m\to\infty, n\to\infty$ then we say that $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{m,n}$ is summable to S by (C,1,1) method.

Consider the double infinite series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}$

in this case,

$$\begin{aligned} S_{m,n} &= \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j} \\ &=\left(\sum_{i=0}^{m}(-1)^{(i)}\right)\left(\sum_{j=0}^{n}(-1)^{(j)}\right) \\ &=(1-1+1-\cdots+(-1)^m)(1-1+1-\cdots+(-1)^n) \\ &=\left(\frac{1-(-1)^{m+1}}{1+1}\right)\left(\frac{1-(-1)^{n+1}}{1+1}\right) \\ &=\frac{1}{4}(1+(-1)^m)(1+(-1)^n) \\ &=\begin{cases} 1, if, m=2n_1, n=2n_2 \\ 0, if, m=2n_1, n=2n_2+1 \\ 0, if, m=2n_1+1, n=2n_2 \\ 0, if, m=2n_1+1, n=2n_2+1 \end{cases} \quad where\ n_1, n_2 \epsilon \mathbb{N}_0. \end{aligned}$$

Then $\lim_{m,n\to\infty,\infty} S_{m,n}$ does not exist. The double infinite series$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{(m+n)}$ is not convergent.

Let us consider $\sigma_{m,n.}$

$$\sigma_{m,n.} = \sum_{i=0}^{m}\sum_{j=0}^{n}\left(1-\frac{i}{m+1}\right)\left(1-\frac{j}{n+1}\right)(-1)^{(i+j)}$$

$$=\frac{(3+(-1)^m+2m)(3+(-1)^n+2n)}{16(m+1)(n+1)}$$

then

$$\sigma_{m,n.}\to\frac{1}{4} \text{ as } m\to\infty, n\to\infty.$$

Therefore the series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{(m+n)}$, is summable to $\frac{1}{4}$ by(C,1,1) method.

2.5 Generalized Carleson operator

Write

$$(S_{N,N}f)(x) = \sum_{n=0}^{N}\sum_{k=-N}^{N}\langle f, w_n(.-k)\rangle w_n(x-k), f\in L^p(\mathbb{R}),\ 1<p<\infty,$$

$$(\sigma_{N,N}f)(x) = \frac{1}{(N+1)^2}\sum_{i=0}^{N}\sum_{j=0}^{N}(S_{i,j}f)(x)$$
$$=\sum_{n=0}^{N}\sum_{k=-N}^{N}\left(1-\frac{n}{N+1}\right)\left(1-\frac{|k|}{N+1}\right)\langle f, w_n(.-k)\rangle\, w_n(x-k)$$

The Carleson operator for the Walsh type wavelet packet system, denoted by L, is defined by

$$(Lf)(x) = \sup_{N\geq 1}\left|\sum_{n=0}^{N}\sum_{k=-N}^{N}\langle f, w_n(.-k)\rangle\, w_n(x-k)\right|.\ f\in L^p(\mathbb{R}),\ 1<p<\infty,$$
$$= \sup_{N\geq 1}\left|(S_{N,N}f)(x)\right|.$$

The generalized carleson operator for the Walsh type wavelet packet system denoted by L_c is defined by

$$(L_c\mathrm{f})(x) = \sup_{N\geq 1}\left|\frac{1}{(n+1)^2}\sum_{i=0}^{N}\sum_{j=0}^{N}(S_{i,j}f)(x)\right|$$
$$= \sup_{N\geq 1}\left|(S_{N,N}f)(x)\right|.$$
$$= \sup_{N\geq 1}\left|\sum_{n=0}^{N}\sum_{k=-N}^{N}\left(1-\frac{n}{N+1}\right)\left(1-\frac{|k|}{N+1}\right)\langle f, w_n(.-k)\rangle\, w_n(x-k)\right| \quad (2.6).$$

Let us define the generalized Carleson Operator for the periodic Walsh type wavelet packet system $\{\widetilde{w}_n\}$.
Write $(s_N f)(x) = \sum_{n=0}^{N}\langle f,\widetilde{w}_n\rangle\widetilde{w}_n(x), f\in L^p(\mathbb{R})$, $1<p<\infty$,

$$(\sigma_N f)(x) = \frac{1}{(N+1)}\sum_{n=0}^{N}(s_N f)(x)$$
$$= \frac{1}{(N+1)}\sum_{n=0}^{N}\sum_{i=0}^{n}\langle f,\widetilde{w}_n\rangle(\widetilde{w}_i(x))$$
$$= \sum_{n=0}^{N}(1-\frac{n}{N+1}\langle f,\widetilde{w}_n\rangle\widetilde{w}_n(x).$$

The Carleson operator for the periodic Walsh type wavelet packet system, denoted by $\mathbb{G}$, is defined by

$$(\mathbb{G}f)(x) = \sup_{N\geq 1}\left|\sum_{n=0}^{N}\langle f,\widetilde{w}_n\rangle\widetilde{w}_n(x)\right|, f\in L^p(\mathbb{R}),\ 1<p<\infty,$$
$$= \sup_{N\geq 1}\left|(s_N f)(x)\right|.$$

The generalized Carleson operator for the periodic Walsh type wavelet type packet system, denoted by , is defined by

$$(\mathbb{G}_c f)(x) = \sup_{N\geq 1}\left|\frac{(s_0 f)(x)+(s_1 f)(x)+(s_2 f)(x)+\cdots+(s_N f)(x)}{N+1}\right|$$
$$= \sup_{N\geq 1}\left|\frac{1}{N+1}\sum_{n=0}^{N}(s_n f)(x)\right|$$
$$= \sup_{N\geq 1}\left|(\sigma_N f)(x)\right|$$
$$= \sup_{N\geq 1}\left|\sum_{n=0}^{N}\left(1-\frac{n}{N+1}\right)\langle f,\widetilde{w}_n\rangle\widetilde{w}_n(x)\right|. \qquad (2.7)$$

2.6 Strong type (p,p) Operator

An operator T defined on $L^p(\mathbb{R})$ is of strong type(p,p) if it is sub-linear and there is a constant C such that$\|Tf\|_p \leq C\|f\|_p$ for all $f\in L^p(\mathbb{R})$.

III. Main Results

In this paper, two new Theorem have been established in the following form:

Theorem 3.1 Let $\{w_n\}$ be a family of Walsh-type wavelet packet system. Then the generalized Carleson operator L_c defined by (2,6) for any Walsh-type wavelet packet system is of strong type(p,p),1$< p<\infty$.

Theorem 3.2 Let $\{\widetilde{w}_n\}$ be a periodic Walsh-type wavelet packets. Then the generalized Carleson Operator $\mathbb{G}_c$ defined by (2.7) for periodic Walsh type wavelet packet expansion of any $f\in L^p(\mathbb{R}), 1<p<\infty$, converges a.e...

3.1 Lemmas

For the proof of the theorems following lemmas are required:

Lemma 3.1 (Zygmund[12],p.197),If$v_1, v_2, v_3, \ldots v_n$ are non negative and non increasing,then

$$|u_1v_1+u_2v_2+u_3v_3+\cdots+u_nv_n| \leq v_1 max|U_k|$$

Where
$U_k = u_1+u_2+u_3+\cdots+u_k$ for k=1,2,3,...,n.

Lemma 3.2 Let $f_1\in L^2(\mathbb{R})$, and define $\{f_n\}_{n\geq 2}$ recursively by

$$f_n(x) = \begin{cases} f_m(2x) + f_m(2x-1), n = 2m \\ f_m(2x) - f_m(2x-1), n = 2m+1 \end{cases}$$

Then

$$f_m(x) = \sum_{p=0}^{2^j-1} W_{m-2^j p 2^{-j}} f_1(2^j x\text{-p})$$

for $m, j \in \mathbb{N}, 2^j \leq m < 2^{j+1}$.

Lemma 3.3 Let $\{w_n\}_{n\geq 0}$ be a family of Walsh type wavelet packets. If $w_1 \in C^1(\mathbb{R})$ then there exists an isomorphism $\psi : L^p(\mathbb{R}) \to L^p(\mathbb{R}), 1 < p < \infty, such\ that$

$$\psi w_{n(.-k)} = w_n(.-k), n \geq 0, k \in \mathbb{Z}.$$

Lemma (3.2) and (3.3) can be easily proved.

Lemma 3.4 (Billard [1] and Sjölin [9].

Let $f \in L^1[0,1)$ and define

$$S_n(x,f) = \sum_{k=0}^{n} \int_0^1 f(t) W_k(t) dt W_k(x).$$

Then the Carleson operator G defined by

$$(Gf)(x) = \sup_n |S_n(x,f)|$$

is of strong type (p,p) for $1 < p < \infty$.

3.2 Proof of Theorem 3.1

For, $f, g, L^p(\mathbb{R})$

$$\big(L_c(f+g)\big)(x) = \sup_{N \geq 1} \left|\sigma_{N,N}(f+g)(x)\right|$$

$$= \sup_{N \geq 1} \left|\sum_{n=0}^{N} \sum_{k=-N}^{N} \left(1 - \frac{n}{N+1}\right)\left(1 - \frac{|k|}{N+1}\right) \langle f+g, w_n(.-k)\rangle w_n(x-k)\right|$$

$$= \sup_{N \geq 1} \left|\sum_{n=0}^{N} \sum_{k=-N}^{N} \left(1 - \frac{n}{N+1}\right)\left(1 - \frac{|k|}{N+1}\right) \langle f, w_n(.-k)\rangle + \langle g, w_n(.-k)\rangle w_n(x-k)\right|$$

$$\leq \sup_{N \geq 1} \left|\sum_{n=0}^{N} \sum_{k=-N}^{N} \left(1 - \frac{n}{N+1}\right)\left(1 - \frac{|k|}{N+1}\right) \langle f, w_n(.-k)\rangle w_n(x-k)\right|$$

$$+ \sup_{N \geq 1} \left|\sum_{n=0}^{N} \sum_{k=-N}^{N} \left(1 - \frac{n}{N+1}\right)\left(1 - \frac{|k|}{N+1}\right) \langle g, w_n(.-k)\rangle w_n(x-k)\right|$$

$$= \sup_{N \geq 1} \left|(\sigma_{N,N} f)(x)\right| + \sup_{N \geq 1} \left|(\sigma_{N,N} g)(x)\right|$$

$$= (L_c f)(x) + (L_c g)(x).$$

Also for $\alpha \in \mathbb{R}$

$$(L_c \alpha f)(x) = \sup_{N \geq 1} \left|(\sigma_{N,N}(\alpha f))(x)\right|$$

$$= \sup_{N \geq 1} \left|\sum_{n=0}^{N} \sum_{k=-N}^{N} \left(1 - \frac{n}{N+1}\right)\left(1 - \frac{|k|}{N+1}\right) \langle \alpha f, w_n(.-k)\rangle w_n(x-k)\right|$$

$$= |\alpha| \sup_{N \geq 1} \left|\sum_{n=0}^{N} \sum_{k=-N}^{N} \left(1 - \frac{n}{N+1}\right)\left(1 - \frac{|k|}{N+1}\right) \langle f, w_n(.-k)\rangle w_n(x-k)\right|$$

$$= |\alpha| (L_c f)(x).$$

Choose $M \in \mathbb{N}$ such that

$\text{Supp}(w_n) \subset [-M, M]$ for $n \geq 0$. Fix $p \in (1, \infty)$ and take any $f(x) = \sum_{n\geq 0, k\in\mathbb{Z}} \langle f, w_n(.-k)\rangle\, w_n(x-k) \in L^p(\mathbb{R})$. Define

$$f_k(x) = \sum_{n\geq 0, k\in\mathbb{Z}} \langle f, w_n(.-k)\rangle\, w_n(x-k), g_k(x) = \sum_{n\geq 0, k\in\mathbb{Z}} \langle f, w_n(.-k)\rangle\, W_n(x-k).$$

We have $\|f_k\|_p \approx \|g_k\|_p, with$ bounds independent of k (Lemma 3.3) .Note that for $|\{x \in [l, l+1) : |L_c f(x)| > \alpha\}| \leq \frac{C}{\alpha^p} \sum_{k=l-n}^{l+1+n} \int |L_c f_k(x)|^p\, dx$. Using the Marcinkiewiez interpolation theorem, it suffices to prove that

$$\|L_c f_k\|_p \leq C \|f_k\|_p,$$

where C is a constant independent of k.

Since

$$\sum_{l\in\mathbb{Z}}\sum_{k=l-n}^{l+1+n}\|f_k\|_p^p \le 2(n+1)\sum_{k\in\mathbb{Z}}\|f_k\|_p^p \le 2C(n+1)\sum_{k\in\mathbb{Z}}\|g_k\|_p^p \le C_1\|f\|_p^p$$

where C_1 is constant.

Without loss of generality, we assume that k=0. Let K $\in \mathbb{N}$ be the scale from which only the Haar filter is used to generate the wavelet packets $(w_n)_{n\ge 2^{k+1}}$.
Let N $\in \mathbb{N}$ and suppose $2^j \le N \le 2^{J+1}$ for some J$> K+1. Clearly, for$ each $x \in \mathbb{R}$.

$$\begin{aligned}(L_c f)(x) &= \sup_{N\ge 1}\left|\sum_{n=0}^{N}\sum_{k=-N}^{N}\left(1-\frac{n}{N+1}\right)\left(1-\frac{|k|}{N+1}\right)\langle f, w_n(.-k)\rangle w_n(x-k)\right|\\
&= \sup_{N\ge 1}\left|\sum_{n=0}^{N}\left(1-\frac{n}{N+1}\right)\langle f, w_n(.)\rangle\, w_n(x)\right|, \quad k=0,\\
&\le \sup_{1\le N<2^{k+1}}\left|\sum_{n=0}^{N}\left(1-\frac{n}{N+1}\right)\langle f, w_n\,(.)\rangle\, w_n(x)\right|\\
&+\sup_{J>K+1}\left|\sum_{n=2^{K+1}}^{2^J-1}\left(1-\frac{n}{N+1}\right)\langle f, w_n\,(.)\rangle\, w_n\,(x)\right|\\
&+\sup_{J>K+1}\left\{\sup_{2^J\le N<2^{J+1}}\left|\sum_{n=2^J}^{N}\left(1-\frac{n}{N-2^J+1}\right)\langle f, w_n(.)\rangle\, w_n\,(x)\right|\right\}\\
&= J_1+J_2+J_3 \qquad \text{say.}\end{aligned} \tag{3.2}$$

Using Lemma 3.1, we have

$$\begin{aligned}J_1 &= \sup_{1\le N<2^{k+1}}\left|\sum_{n=0}^{2^{k+1}-1}\left(1-\frac{n}{N+1}\right)\langle f, w_n\,(.)\rangle\, w_n(x)\right|\\
&\le \sup_{1\le N<2^{k+1}} \max\sum_{0=n\le 2^{K+1}-1}\langle f, w_n\,(.)\rangle\, w_n(x)\\
&\le \sum_{n=0}^{2^{K+1}-1}|\langle f, w_n(.)\rangle|\ |w_n\,(x)|\\
&\le |\langle f, w_n(.)\rangle|\ \|w_n(x)\|_\infty\, \chi_{[-N,N]}\,(x)\\
&\le \|f_0\|_p\ \sum_{n=0}^{2^{K+1}-1}\|w_n\|_q\ \ \|w_n\,(x)\|_\infty\, \chi_{[-N,N]}\,(x).\end{aligned} \tag{3.3}$$

Next,

$$\begin{aligned}J_2 &= \sup_{J>K+1}\left|\sum_{n=2^{K+1}}^{2^J-1}\left(1-\frac{n}{N+1}\right)\langle f, w_n(.)\rangle w_n(x)\right|\\
&\le \sup_{J>k+1}\max\left|\sum_{2^{k+1}=n\le 2^J-1}\langle f, w_n(.)\rangle w_n(x)\right|.\end{aligned}$$

Also

$$\begin{aligned}\left\|\sup_{J>k+1}\sum_{n=2^{k+1}}^{2^J-1}\left(1-\frac{n}{N+1}\right)\langle f, w_n(.)\rangle w_n(x)\right\|_p &\le \sup_{J>k+1}\max\sum_{2^{k+1}=n\le 2^J-1}|f, w_n(.)|\,\|w_n(x)\|_p\\
&\le C\sum_{n=0}^{\infty}\langle f, w_n(.)\rangle\|w_n(x)\|_p\\
&= C\|f_0\|_p.\end{aligned} \tag{3.4}$$

Consider J_3,

$$\begin{aligned}J_3 &= \sup_{2^J\le N<2^J+1}\left|\sum_{n=2^J}^{N}\left(1-\frac{n}{N+1}\right)\langle f, w_n(.)\rangle w_n(x)\right|\\
&\le \sum_{j=0}^{2^K-1}\left(\sup_{2^J+j2^{J-K}\le N<2^J+(j+1)2^{J-K}}\left|\sum_{n=2^J+j2^{J-K}}^{N}\left(1-\frac{n}{N+1}\right)\langle f, w_n(.)\rangle w_n(x)\right|\right),\end{aligned}$$

so it suffices to prove that

$$\left\|\sup_{j>k+1}\left\{\sup_{2^J+j2^{J-K}\le N<2^J+(j+1)2^{J-K}}\left|\sum_{n=2^J+j2^{J-K}}^{N}\left(1-\frac{n}{N+1}\right)\langle f, w_n(.)\rangle w_n(x)\right|\right\}\right\|_p \le C\|f_0\|_p$$

for $j=0,1,2,\dots,2^k-1$. Fix $J>K+1$, $0\le j\le 2^{2^k-1}$ and $2^J+j2^{J_K}\le N<2^J+(j+1)2^{J-K}$.

Write,

$$j'_3 = \sum_{n=2^J+j2^{J-K}}^{N} (1-\frac{n}{N+1})\langle f, w_n(.)\rangle w_n(x).$$

Using lemma 3.2, we have

$$|j'_3| = \left| \sum_{s=0}^{2^{J-K}-1} \left\{ \sum_{n=2^J+j2^{J-K}}^{N} (1-\frac{n}{N+1})\langle f, w_n(.)\rangle W_{n-2^J-j2^{J-K(s2^{-(J-K)})}} \right\} w_2k + j^{(2^{J-K}x-s)} \right|.$$

Define

$$F_N(t) = \sum_{n=2^J+j2^{J-K}}^{N} \left(1-\frac{n}{N+1}\right)\langle f, w_n(.)\rangle W_{n-2^J-j2^{J-K},}(t),$$

and

$$F(t) = \sup_{N<2^J+(j+1)2^{J-K}} |F_N(t)|.$$

We have

$$|j'_3| = \left| \sum_{n=2^J+j2^{J-K}}^{N} \left(1-\frac{n}{N+1}\right)\langle f, w_n(.)\rangle w_n(x) \right|,$$

$$\leq max \left| \sum_{2^J+j2^{J-K}=n<N} \langle f, w_n(.)\rangle w_n(x) \right|,$$

$$\leq \sum_{S=0}^{2^{J-K}-1} F(s2^{-(J-K)}) \left| w_{2^k+j((2^{J-K})x-s)} \right|,$$

and using the compact support of the wavelet packets,

$$|j'_3| = \left| \sum_{n=2^J+j2^{J-K}}^{N} \left(1-\frac{n}{N+1}\right)\langle f, w_n(.)\rangle w_n(x) \right|,$$

$$\leq max \left| \sum_{2^J+j2^{J-K}=n<N} \langle f, w_n(.)\rangle w_n(x) \right|$$

$$\leq \|w_2k+j\|_\infty \sum_{l=-M}^{M+1} F(([2^{J-K}x]+l)2^{-(J-K)}).$$

Note that F is constant on dyadic interval of type $[l2^{-(j-k)}, (l+1)2^{-(j-k))}$ and taking $\Delta_l = (([2^{J-K}x+l]2^{-(J-K)}, [2^{J-K}x+(l+1)]2^{-(J-K)})$,
we have

$$|j'_3| = \left| \sum_{n=2^J+j2^{J-K}}^{N} \left(1-\frac{n}{N+1}\right)\langle f, w_n(.)\rangle w_n(x) \right|$$

$$\leq max \left| \sum_{2^J+j2^{J-K}=n<N}^{N} \langle f, w_n(.)\rangle w_n(x) \right|$$

$$\leq \|w_2k+j\|_\infty \sum_{l=-M}^{M+1} |\Delta_l|^{-1} \int_{\Delta_l} F(t)dt.$$

We need an estimate of F that does not depend on J . Note that for k, $0\leq k < 2^{J-K}$, using (2.3)

$$W_{2^J+j2^{J-K}}(t)W_k(t) = W_{2^J+j2^{J-K}+k}(t),$$

since the binary expansions of 2^J+j2^{J-K} and of k have no 1's in common.
Hence,

$$|F_N(t)| = \left|W_{2^J+j2^{J-K}}(t)F_N(t)\right|$$

$$= \left| \sum_{n=2^J+j2^{J-K}}^{N} \left(1-\frac{n}{N+1}\right)\langle f, w_n(.)\rangle W_n(t) \right|$$

$$\leq max\left|\sum_{2^J+j2^{J-K}=n<N}^{N}\langle f,w_n(.)\rangle W_n(t)\right|.$$

Using lemma (3.4) , we have F(t)$\leq 2(Ggo)(t)$. Thus,

$$\left|\sum_{n=2^J+j2^{J-K}}^{N}\left(1-\frac{n}{N+1}\right)\langle f,w_n(.)\rangle w_n(x)\right|$$
$$\leq max\left|\sum_{2^J+j2^{J-K}=n<N}^{N}\langle f,w_n(.)\rangle w_n(x)\right|$$
$$\leq 2\|w_2k+j\|_\infty\sum_{l=-M}^{M+1}|\Delta_l|^{-1}\int_{\Delta_l}(Gg_0)(t)dt.$$

Let Δ_l^* be the smallest dyadic interval containing Δ_l and x , and note that $|\Delta_l^*|\leq(M+1)\ |\Delta_l|$ since $x\in(\Delta_0-D)$.

We have

$$\left|\sum_{n=2^J+j2^{J-K}}^{N}\left(1-\frac{n}{N+1}\right)\langle f,w_n(.)\rangle w_n(x)\right|$$
$$\leq max\left|\sum_{2^J+j2^{J-K}=n<N}\langle f,w_n(.)\rangle w_n(x)\right|$$
$$\leq 2\|w_2k+j\|_\infty\sum_{l=-M}^{M+1}|\Delta_l|^{-1}\int_{\Delta_l}(Gg_0)(t)dt$$
$$\leq 4\|w_2k+j\|_\infty(M+1)^2\big(M^*(Gg_0)(x)\big) \tag{3.5}$$

Where M^* is the maximal operator of Hardy and Little wood . The right hand side of (3.5) neither depend on N nor J so we may conclude that

$$J_3\leq \sup_{J>K+1}\sum_{j=0}^{2^K-1}\left(\sup_{2^J+j2^{J-K}\leq N<2^J+(j+1)2^{J-K}}\left|\sum_{n=2^J+j2^{J-K}}^{N}(1-\frac{n}{N+1})\langle f,w_n(.)\rangle w_n(x)\right|\right)$$
$$\leq\sum_{j=0}^{2^k-1}\begin{pmatrix}\sup\\ 2^J+j2^{J-K}\leq N<2^J+(j+1)2^{J-K}\\ \left(max\left|\sum_{2^J+j2^{J-K}=n<N}\langle f,w_n(.)\rangle w_n(x)\right|\right)\end{pmatrix}$$
$$\leq 4\|w_2k+j\|_\infty(M+1)^2\big(M^*(Gg_0)(x)\big),\qquad a.e.. \tag{3.6}$$

Using (Sjӧlin [9]), M^* and G both of strong type (p,p), hence both are bounded.

$$\left\|\sup_{J>K+1}J_3\right\|_p\leq\left\|\sup_{J>K+1}\sum_{j=0}^{2^K-1}\left\{\sup_{2^J+j2^{J-K}\leq N<2^J+(j+1)2^{J-K}}\left|\sum_{n=2^J+j2^{J-k}}^{N}(1-\frac{n}{N+1})\langle f,w_n(.)\rangle w_n(x)\right|\right\}\right\|_p$$
$$\leq C\|g_0\|_p$$
$$\leq C_1\|f_0\|_p,\qquad j=0,1,2,3,\ldots,2^K\text{-}1.$$

Thus the Theorem 3.1 is completely established.

3.3 proof of the Theorem 3.2

Let $f\in L^p[0,1)$, and choose M$\in\mathbb{N}$ such that supp(w_n)$\subset[-M,M]$ for n≥ 0. Then

$$(\mathbb{G}_cf)(x)=\sup_{N\geq 1}\left|\sum_{n=0}^{N}\left(1-\frac{n}{N+1}\right)\langle f,\widetilde{w}_n\rangle\widetilde{w}_n(x)\right|$$
$$=\sup_{N\geq 1}\left|\sum_{n=0}^{N}\left(1-\frac{n}{N+1}\right)\langle f,\sum_{k_1=-M}^{M+1}w_n(.-k_1)\rangle\sum_{k_2=-M}^{M+1}w_n(x-k_2)\right|$$

$$=\sup_{N\geq 1}\left|\sum_{n=0}^{N}\left(1-\frac{n}{N+1}\right)\sum_{k_1=-M}^{M+1}\langle f,w_n(.-k_1)\rangle\sum_{k_2=-M}^{M+1}w_n(x-k_2)\right|$$
$$=\sup_{N\geq 1}\left|\sum_{k_1=-M}^{M+1}\sum_{k_2=-M}^{M+1}\left(\sum_{n=0}^{N}\left(1-\frac{n}{N+1}\right)\langle f,w_n(.-k_1)\rangle\, w_n(x-k_2)\right)\right|$$
$$=\sup_{N\geq 1}\left|\sum_{k_1=-M}^{M+1}\sum_{k_2=-M}^{M+1}\sum_{n=0}^{N}\left(1-\frac{n}{N+1}\right)\int_0^1 f(y)\overline{w_n(y-k_1)}\,dy w_n(x-k_2)\right|$$

Following the proof of theorem 3.1, it can be proved that the generalized Carleson operator $\mathbb{G}_c$ for the periodic Walsh type wavelet packet expansions converges a.e..

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